# Comments on elliptic genera and 3d gravity

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A lot of my talk will be introductory or general; any new material is based on a paper to appear with:

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# I. Introduction and motivation

AdS/CFT gives us our most concrete definition of non-perturbative quantum gravity.

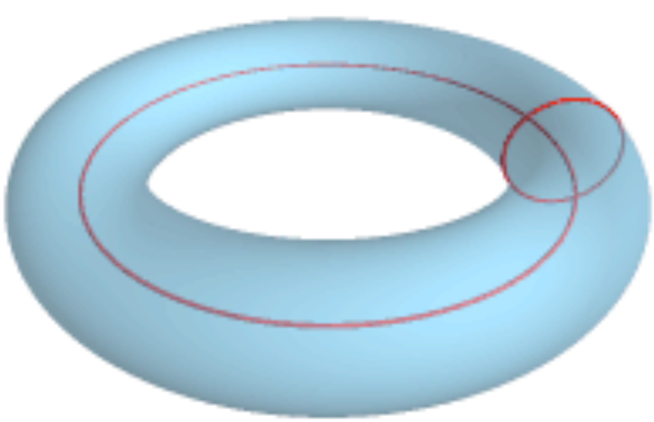
In general, and in particular in 2d, CFTs are relatively well understood (compared to quantum gravity).

Natural question: which 2d CFTs admit large-radius gravity duals?

There have been interesting recent works aimed at clarifying this question.

Hartman, Keller, Stoica

In general, a 2d CFT has a torus partition function which is modular invariant:



$$Z(q,\bar{q}) = \mathrm{Tr}\left(q^{L_0}\bar{q}^{\bar{L}_0}\right), \quad q = e^{2\pi i\tau}$$

However, it is in general difficult to compute. And if the CFT comes with exactly marginal operators, Z of course depends sensitively on the point in moduli space one chooses.

Is there a cruder measure which is:

a) calculable

b) can tell us if the CFT admits a gravity description anywhere in its moduli space?

We won't succeed in constructing a fool-proof such measure. But we will make some simple observations about one such candidate here, and describe results of computations checking several canonical classes of 2d CFTs.

## II. The elliptic genus

We will sacrifice some generality by considering 2d theories with some amount of supersymmetry.

This is advantageous because it allows one to define supersymmetric indices.

The canonical example is the Witten index.

Consider a supersymmetric quantum mechanics theory with a supercharge satisfying

$$Q^2 = 0, \quad \{Q, Q^\dagger\} = H$$

Assume the theory also has a fermion # symmetry, and Q is odd.

Then one can easily prove two powerful statements:

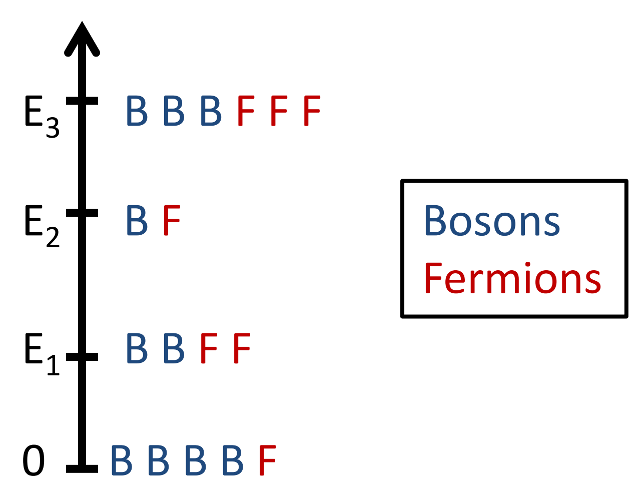
-- all states have non-negative energy

-- states at positive energy are paired by the action of Q

Now, one can define an index:

- The Witten index

$$Z_{\text{Witten}} = \mathrm{Tr}(-1)^F = n_B - n_F = \mathrm{Tr}\left((-1)^F q^H\right)$$



(Note: I avoid here, and later, discussing subtleties that can arise when the spectrum is not discrete. These are important in appearance of e.g. mock modular forms in physics.)

The Witten index is just a number. A quantity with more information -- an entire q-series -- is available in supersymmetric 2d QFTs.

We'll mostly focus on theories with at least (2,2) supersymmetry.

This means that each chirality has generators

$$T, G^{+}, G^{-}, J \ .$$

Famous examples including Calabi-Yau sigma models, and the Hilbert scheme of N points on a K3 surface, dual to $AdS_3 \times S^3 \times K3$ gravity.

In any such theory we can define the elliptic genus:

$$Z_{\mathrm{EG}}(\tau, z) = \mathrm{Tr}_{RR}\left((-1)^{J_0+F_R} q^{L_0} y^{J_0} \bar{q}^{\bar{L}_0}\right)$$

Unpacking the right-moving stuff, we see it is a right-moving Witten index!

So - in theories with discrete spectrum - this will give us a holomorphic modular object.

In fact, it is what is known as a weak Jacobi form of weight 0 and index c/6.

## III. Facts about weak Jacobi forms

c.f. Dabholkar, Murthy, Zagier

A weak Jacobi form is a holomorphic function on $\mathbb{H}\times\mathbb{C}$ which satisfies:

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^w e^{2\pi i m\frac{cz^2}{c\tau+d}}\phi(\tau,z)\ ,\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})$$

$$\phi(\tau,z+\ell\tau+\ell')=e^{-2\pi i m(\ell^2\tau+2\ell z)}\phi(\tau,z)\ ,\ \ \ell,\ell'\in\mathbb{Z}\ .$$

These invariances imply in particular that we may expand the function as:

$$\phi(\tau, z) = \sum_{n,\ell \in \mathbb{Z}} c(n,\ell) q^n y^\ell,$$

$$c(n,\ell) = (-1)^w c(n,-\ell).$$

Define the polarity of a given term by:

$$D(n,\ell) = \ell^2 - 4mn = -p(n,\ell)$$

This is useful for the following reasons:

1. The theories we consider have spectral flow invariance:

$$L_0 \to L_0 + \theta J_0 + \theta^2 m$$

$$J_0 \to J_0 + 2\theta m$$

This allows one to relate all Fourier coefficients to those with $|l| \leq m$ .

2. One can define the polar part of the Jacobi form: the sum of terms with negative polarity.

3. The full polar part of the form can be determined by just the terms in the polar region $\mathcal{P}_m$:

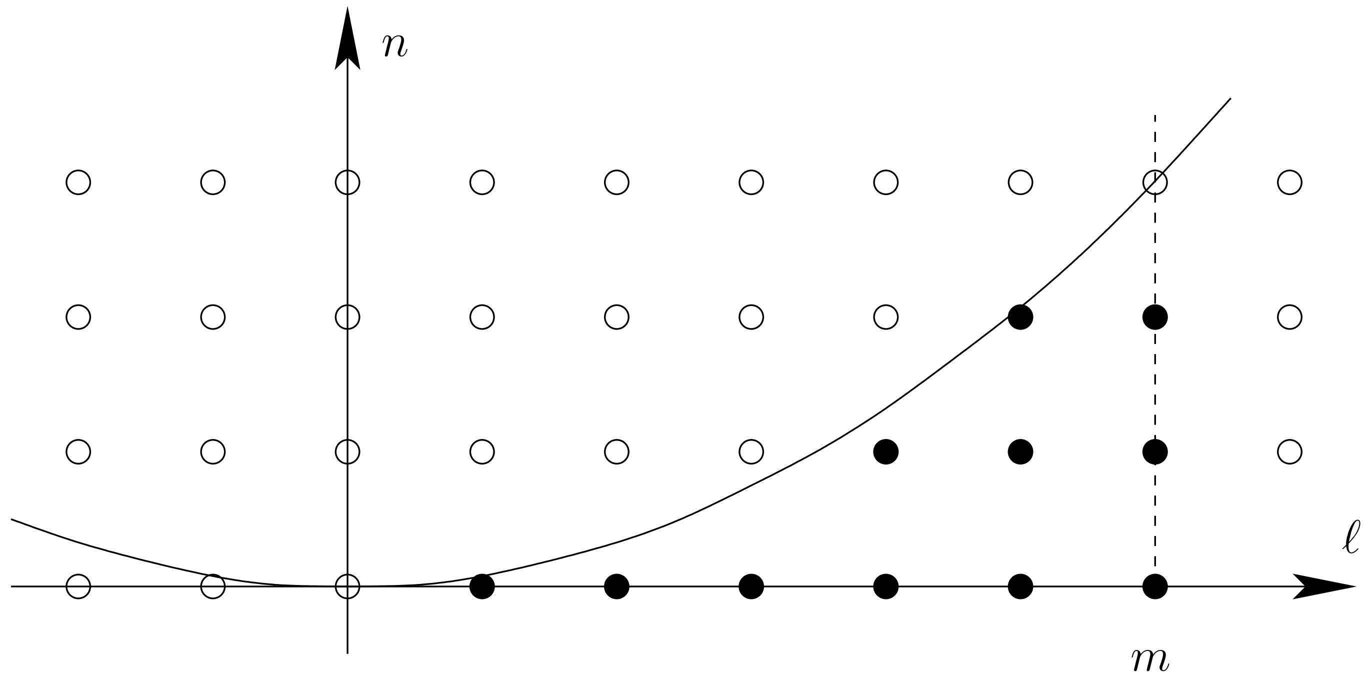



**Figure 1:** A cartoon showing polar states (represented by "•") in the region $\mathcal{P}^{(m)}$. Spectral flow by $\theta = \frac{1}{2}$ relates these states to particle states in the NS sector of an $\mathcal{N} = 2$ superconformal field theory which are holographically dual to particle states in $AdS_3$.

From Gaberdiel, Gukov, Keller, Moore, Ooguri

4. Most importantly, the polar part of the Jacobi form determines the full form.

We now discuss the physics of this, and find a bound on the polar coefficients for theories with gravity duals.

# IV. Phase structure and gravity constraints

So, lets talk about gravity theories in AdS3.

Define the "reduced mass" of a particle state in the gravity theory to be:

$$L_0^{\rm red} = L_0 - \frac{1}{4m} J_0^2 - \frac{m}{4}.$$

(These terms sum to -D/4m.)

* It is known since the "Farey tail" of Dijkgraaf/Maldacena/Moore/Verlinde that the terms with $L_0^{\rm red} < 0$ are the ones which contribute to the polar part of the supergravity partition function.

* In contrast, BTZ black hole states in 3d gravity are those states which are non-polar:

$$4mn - \ell^2 > 0$$

$$S_{BH} = 2\pi\sqrt{mE^{\rm red}} = \pi\sqrt{4mn - \ell^2}$$

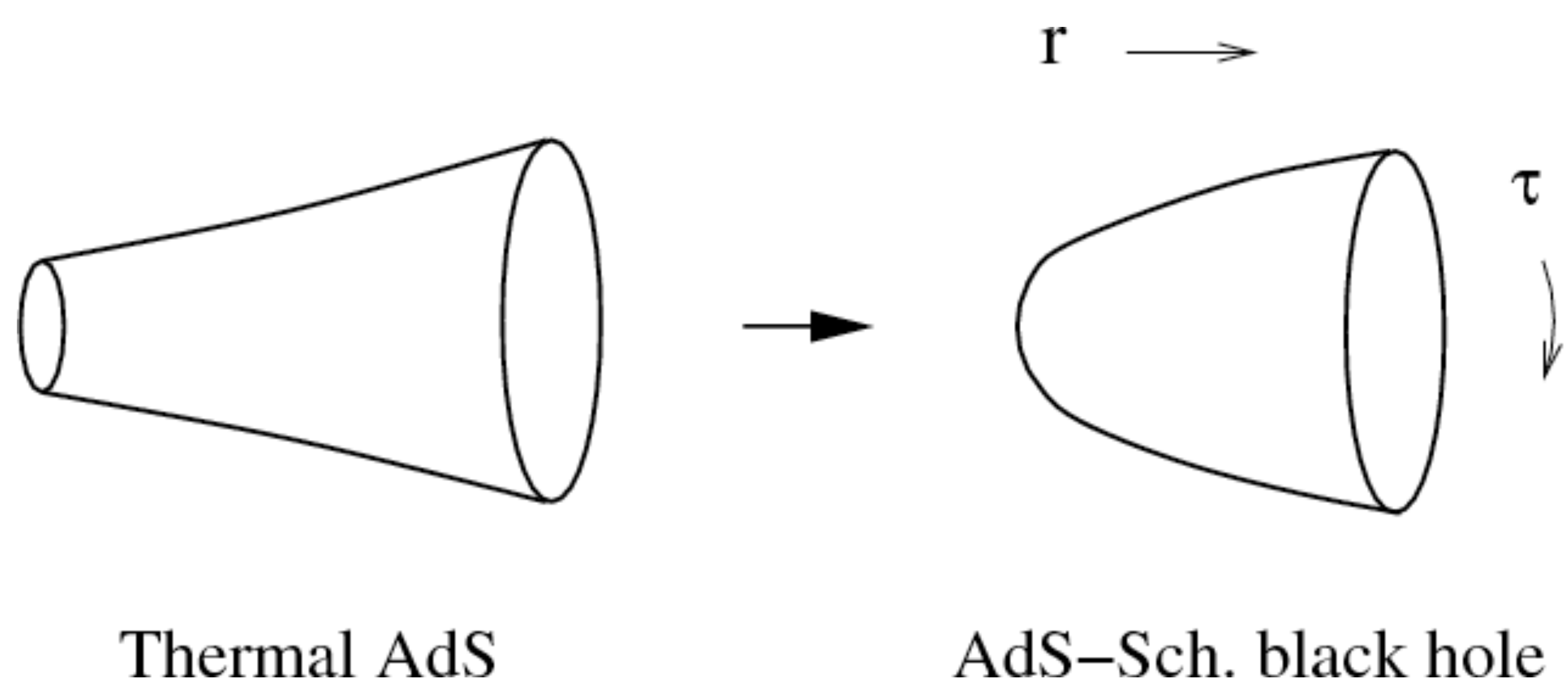
c.f. Cvetic, Larsen '98

SO: the gravity modes contributing to the polar part are precisely modes which are too light to form black holes.

As the counting of these states -- the polar part -- determines the full genus, any bound on genera for theories with gravity duals can be stated in terms of bounds on coefficients of polar terms.

Now, to obtain bounds, one must impose physical criteria.
Two (related) criteria:

1. Known large radius models have a phase structure governed by a Hawking-Page transition:




Low temperatures dominated by "gas of gravitons," high temperatures by black brane geometry.

2. The Bekenstein-Hawking entropy should come out "right" for the black hole states. Cardy only guarantees this for CFT states with

$$\Delta \gg c$$

while we expect in AdS3 gravity, the entropy should come out right when

$$\Delta \sim c \ .$$

A third criterion, well motivated by the formal structure of the problem:

3. Any bound should be spectral flow invariant.

These words can be turned into bounds as follows.

* The low-temperature elliptic genus satisfies:

$$\log Z_{EG,NS}(\tau = i\frac{\beta}{2\pi}) = \frac{c_L}{24}\beta, \quad \beta \gg 2\pi \ .$$

* To keep a free energy of this form until one reaches $\beta = 2\pi$, one then wishes to have:

$$\log Z_{EG,NS}(\tau = i\frac{\beta}{2\pi}) = \frac{c_L}{24}\beta + O(1), \ \ \beta > 2\pi.$$

We can translate this into a condition on polar coefs.

The polar piece of the NS elliptic genus looks like:

$$Z_{ell} = \sum_{(n,\ell):p(n,\ell)<0} c(n,\ell)e^{-\beta d(n,\ell)}$$
$$= e^{-\beta d(0,-m)} \sum_{(n,\ell):p(n,\ell)<0} c(n,\ell)e^{-\beta[d(n,\ell)-d(0,-m)]} \ .$$

Here,

$$d(n,\ell) = \frac{(m+\ell)^2}{4m} - \frac{D(n,\ell)}{4m} = n + \frac{\ell}{2} + \frac{m}{4} \ .$$

Imposing that the excited state contributions do not overwhelm the NS ground state, we see:

$$|c(n,\ell)/c(0,-m)| \leq e^{2\pi[d(n,\ell)-d(0,-m)]} \ .$$

In fact, a spectral flow invariant bound which is stronger than this and guarantees the appearance of the correct black hole entropy is:

In English: energy above NS vacuum.

$$|c(n,\ell)/c(0,-m)| \leq e^{2\pi \frac{[-D(n,\ell)+D(0,-m)]}{4m}}$$

To see that these bounds guarantee the correct entropy, one simply modular transforms:

$$Z(\tau = i\frac{\beta}{2\pi}) = \frac{c}{24}\beta, \quad \beta > 2\pi$$

and finds:

$$\log Z(\tau = i\tfrac{\beta}{2\pi}) = \pi^2 \tfrac{m}{\beta} + \cdots \,, \quad \beta < 2\pi \ .$$

Now, use the standard thermodynamic relations

$$E^{\rm red}(\beta) = -\partial_\beta \log Z = \pi^2 m/\beta^2$$

$$S(\beta) = -(1-\beta\partial_\beta)\log Z = 2\pi^2 m/\beta.$$

One sees immediately that:

$$S(E) = 2\pi\sqrt{mE^{\rm red}}$$

# V. Examples

We now discuss a few examples which satisfy/do not satisfy the kind of bound we derived.

## A. Hilbert scheme of N points on K3

Elliptic genera of symmetric products were discussed extensively in the mid 1990s. Dijkgraaf, Moore, Verlinde, Verlinde

One can define a generating function for elliptic genera:

$$\mathcal{Z}(p,\tau,z) = \sum_{N\geq 0} p^N Z_{EG}(\mathrm{Sym}^N(M);q,y) \ .$$

# Then it is a beautiful fact that:

$$\mathcal{Z}(p,\tau,z) = \prod_{n>0,m\geq 0,l} \frac{1}{(1-p^n q^m y^l)^{c(nm,l)}}.$$

$$Z_{EG}(M;q,y) = \sum_{m\geq 0,l} c(m,l) q^m y^l.$$

We can give some checks that this satisfies our bounds for K3. Consider the terms in the elliptic genus that have vanishing power of q:

$$\mathcal{Z}(p,\tau,z) = \prod_{n>0,l} \frac{1}{(1-p^n y^l)^{c(0,l)}} + O(q)$$

E.g., the most polar term in $\mathrm{Sym}^N$ has the form $y^{-mN}$ where m is the index of the N=1 CFT.

One easily sees that this gets contributions only if one takes the "1" from each term in the product except:

$$\frac{1}{(1-py^{-m})^{c(0,-m)}}$$

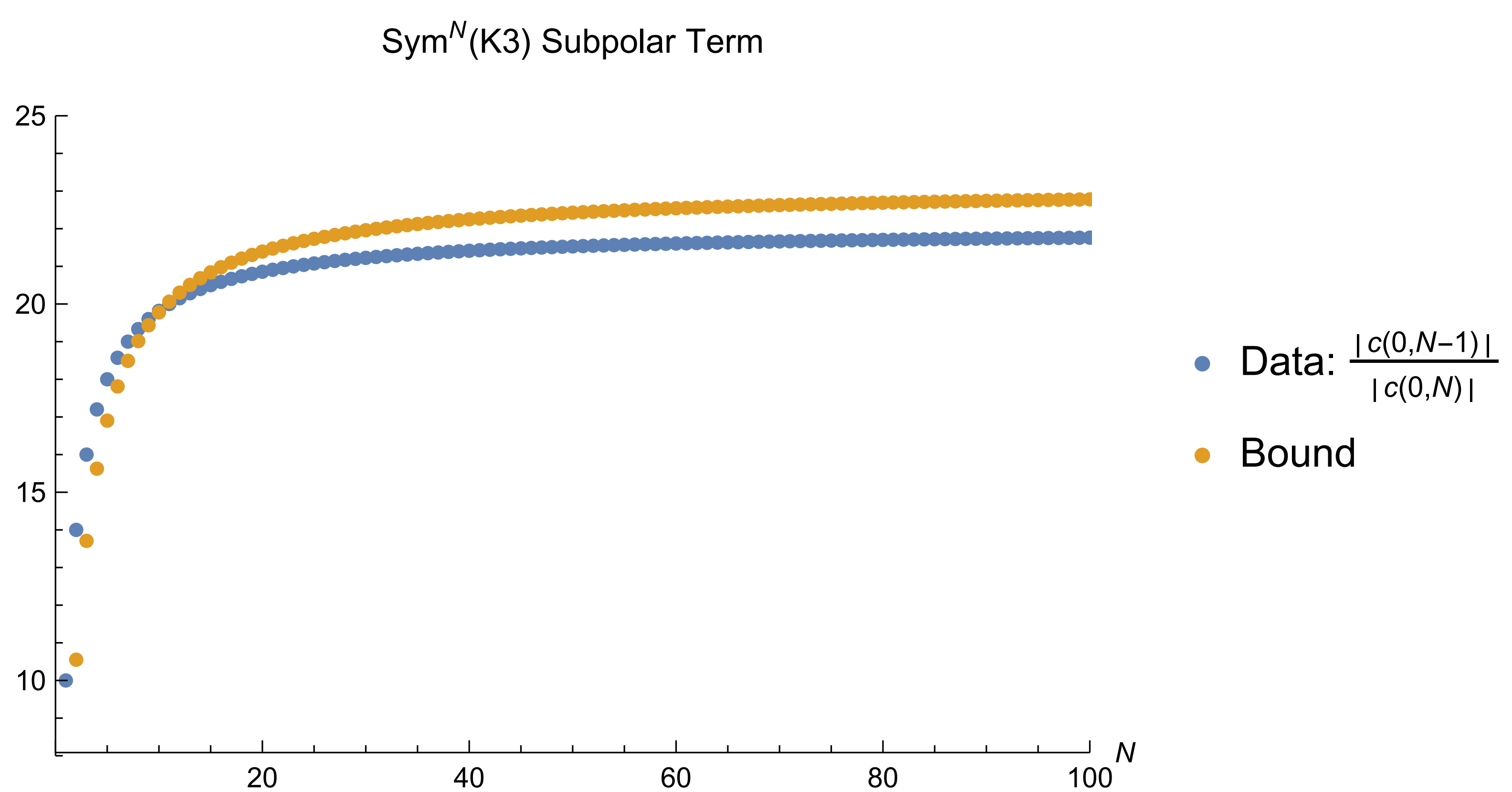
yielding after a moment's thought:

$$c_{\mathrm{Sym}^N M}(0,-Nm) = \binom{c(0,-m)+N-1}{N}.$$

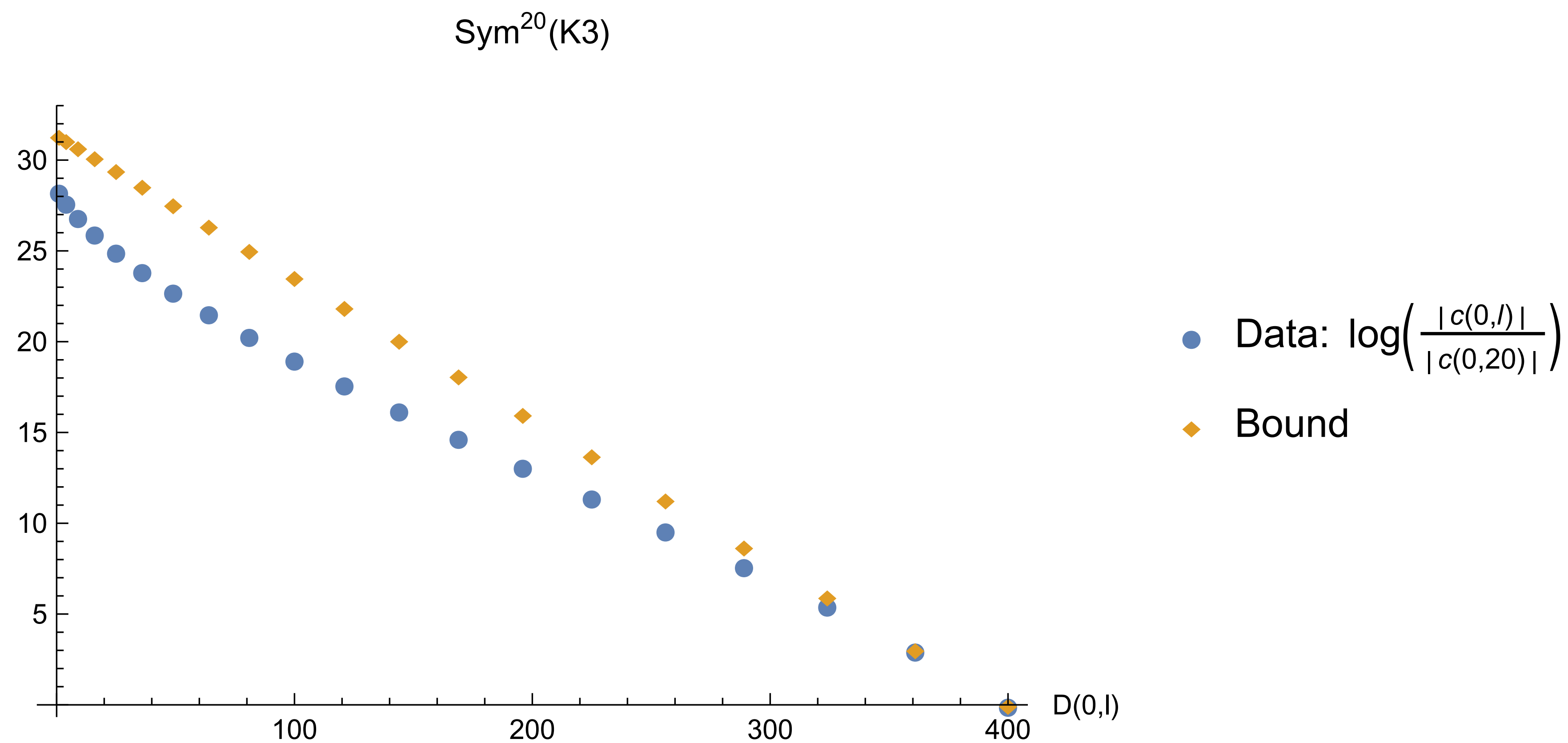
# An obvious generalization works also for the penultimate polar term:

$$c_{\mathrm{Sym}^N M}(0,-Nm+1) = \begin{cases} \binom{c(0,-m)+N-2}{N-1} c(0,-m+1), & \text{if } m>1 \\ \binom{c(0,-m)+N-2}{N-1} c(0,-m+1) + \binom{c(0,-m)+N-3}{N-2} c(0,-m), & \text{if } m=1. \end{cases}$$

# and so forth. Checking against the bound:



# In more detail for one large central charge theory (well, c=120...):




The coefficients of various polarities satisfy bounds that admit simple analytical expressions at large N (not given).

# B. Large N products

The poster-child for not working is a sigma model with target $M^N$ , or more generally such a product CFT.

* The elliptic genus is multiplicative:

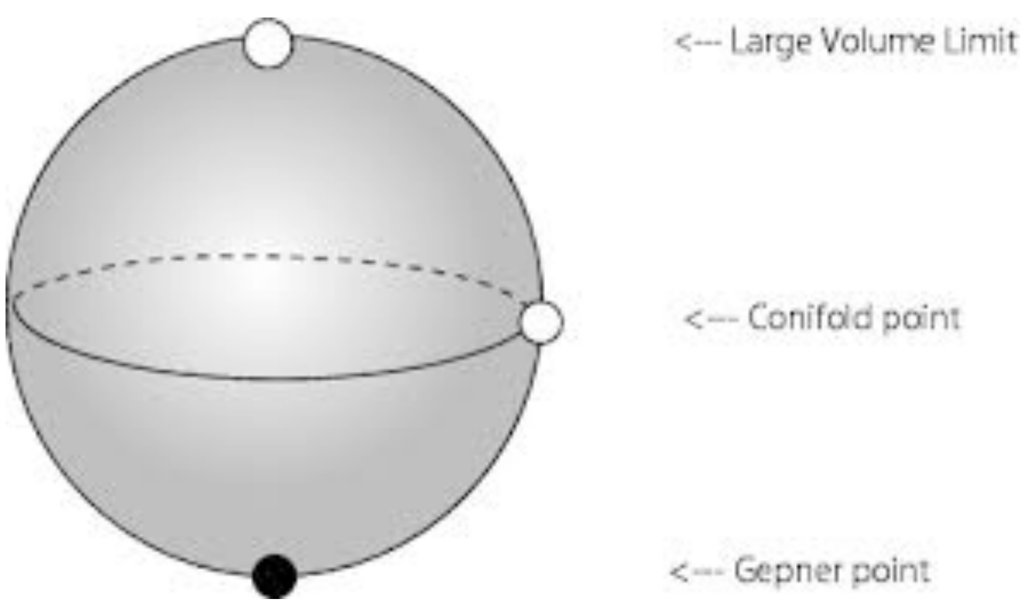
$$Z_{\mathrm{EG}}(M^N) = \left(Z_{\mathrm{EG}}(M)\right)^N \quad .$$

* Suppose the subleading polar term for M is k.

* Then, that for the product is Nk. This (and higher terms) grows with N; the bounds are fixed at large N.

## C. Calabi-Yau spaces of high dimension

It is natural to ask whether some simple Calabi-Yau manifolds of high dimension (but not symmetric products) may satisfy the bound? The simplest family to check is given by hypersurfaces of dimension d in $\mathbb{CP}^{d+1}$.



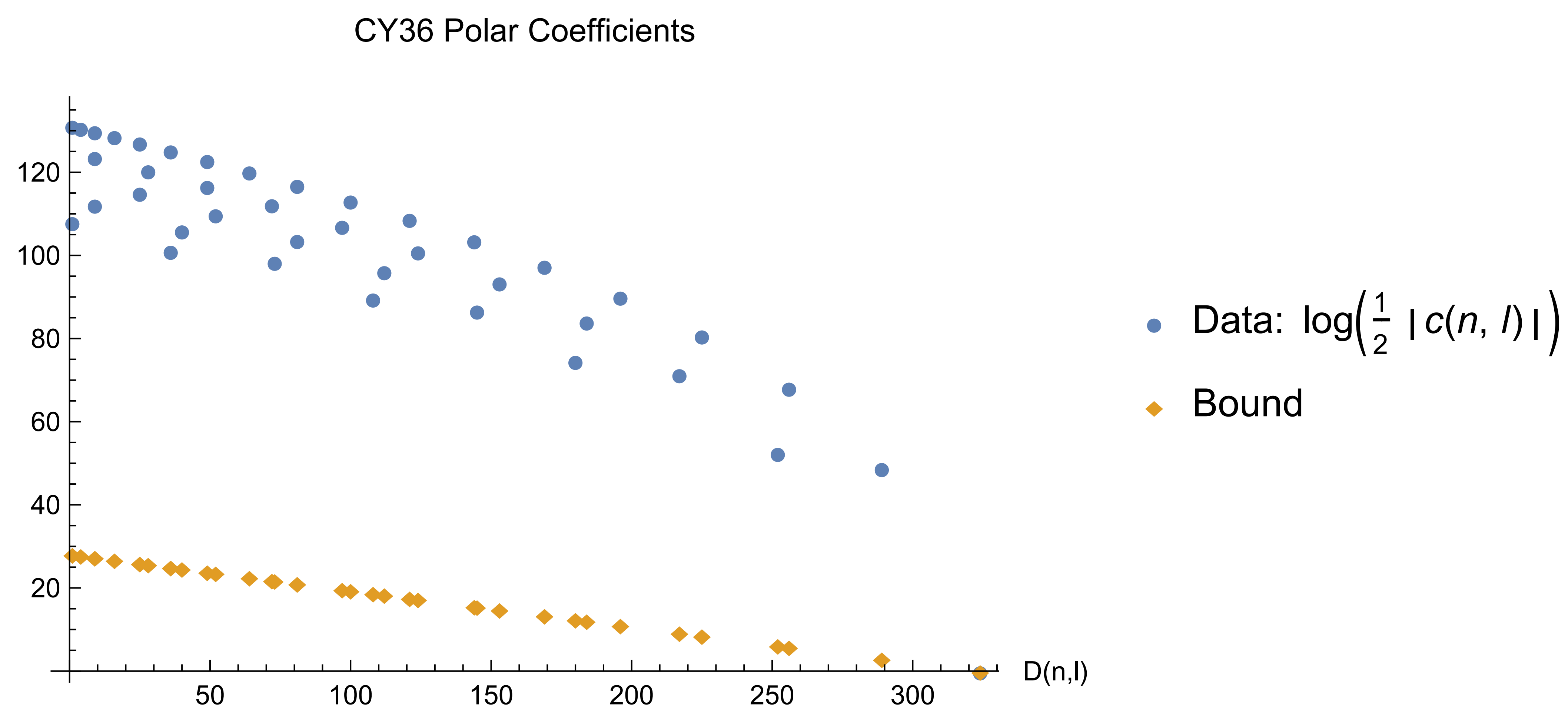

These spaces all admit a soluble Landau-Ginzburg point in moduli space.

# Their elliptic genera were computed in the 1990s:

$$Z^d_{EG}(\tau, z) = \frac{1}{d+2} \sum_{k,\ell=0}^{d+1} y^{-\ell} \frac{\theta_1\left(\tau, -\frac{d+1}{d+2}z + \frac{\ell}{d+2}\tau + \frac{k}{d+2}\right)}{\theta_1\left(\tau, \frac{1}{d+2}z + \frac{\ell}{d+2}\tau + \frac{k}{d+2}\right)}$$

Kawai, Yamada, Yang

# These guys all violate the bound in a spectacular way.



In fact, we can easily understand this analytically. A mathematician's definition of the elliptic genus is:

$$Z_E(\tau,z) = (\sqrt{-1})^{r-D} q^{\frac{r-D}{12}} y^{-\frac{r}{2}} \int_X \mathrm{ch}\Big( \bigotimes_{n=1}^{\infty} \wedge_{-yq^{n-1}} E \otimes \bigotimes_{n=1}^{\infty} \wedge_{-y^{-1}q^n} E^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \Big) \mathrm{td}(X)$$

$$= (\sqrt{-1})^{r-D} q^{\frac{r-D}{12}} y^{-\frac{r}{2}} \Big[ \chi_y(E) + q\big(\textstyle\sum_{s=0}^{r} \{(-y)^{s+1}\chi(\wedge^s E \otimes E) + (-y)^{s-1}\chi(\wedge^s E \otimes E^*) + (-y)^s \chi(\wedge^s E \otimes (T \oplus T^*))\}\big) + \cdots \Big],$$

from which it follows that for a Calabi-Yau manifold,

$$c(0,-m+i) = \sum_k (-1)^{i+k} h^{k,i} \ .$$

Now, deformation theory tells us that for the family of Calabi-Yau spaces under consideration:

$$h^{1,d-1} = \frac{(d+2)\times(d+3)\times\ldots\times(2d+3)}{1\times 2\times\ldots\times(d+2)} - (d+2)^2$$
$$= \binom{2d+3}{d+2} - (d+2)^2.$$

From this it follows immediately that:

$$c_{0,m-1} = \binom{2d+3}{d+2} - (d+2)^2 + 1.$$

Compare to the (large d) bound -- $e^{\pi}$.

# C. The Monster (just for fun)

Thanks to Xi Yin

Can we make a large radius gravity theory with Monster symmetry by considering the symmetric product of the famous FLM orbifold of the Leech lattice?

$$\sum_{N=0}^{\infty} e^{2\pi i N\sigma} Z_{\mathrm{Sym}^N}(\tau) = \frac{e^{-2\pi i\sigma}}{J(\sigma) - J(\tau)}.$$

Borcherds

The Monster CFT does have N=1 supersymmetry; one can define elliptic genera for theories with at least (0,1) SUSY (one gets modular forms under $\Gamma_\theta$ ).

One can prove that for this N-fold symmetric product:

$$q^N Z_{\mathrm{Sym}^N}(\tau) = q^{-1} F(\tau) + \mathcal{O}(q^{N+1}).$$

Here,

$$F(\tau) = \frac{\Delta(\tau)}{E_4(\tau)^2 E_6(\tau)} = \sum_{n=1}^{\infty} a_n q^n.$$

The coefficients can be extracted by contour integral methods. Interestingly:

* Our bounds are satisfied.

* However, in the regime

$$1 \ll n \ll N$$

(where n doesn't depend parametrically on N in any way), the coefficients have growth

$$a_n \sim e^{2\pi n} \; .$$

← Hagedorn growth

Interpretation: This is a large radius gravity theory in Planck units, but the curvature is ~ the string scale. We might instead prefer to focus on theories with:

$$M_{\rm Planck} \gg M_{\rm string} \gg \frac{1}{R_{\rm curvature}}$$

Take away lessons:

* One can associate effectively computable modular objects (determined by polar coefficients), with SCFT2s.

* Simple physical requirements then bound the polar coefficients, leading to constraints that likely only a tiny fraction of all candidate theories satisfy.

* More refined tests distinguishing between low-energy supergravity theories and low-tension string theories (in units of the curvature) are available.